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A modified KdV equation with self-consistent sources in non-uniform media and soliton dynamics

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Abstract

Two non-isospectral modified KdV equations with self-consistent sources are derived, which correspond to the time-dependent spectral parameter λ satisfying $\lambda_t = \lambda$ and $\lambda_t = \lambda^3$, respectively. Gauge transformation between the first non-isospectral equation (corresponding to $\lambda_t = \lambda$) and its isospectral counterpart is given, from which exact solutions and conservation laws for the non-isospectral one are easily listed. Besides, solutions to the two non-isospectral modified KdV equations with self-consistent sources are derived by means of the Hirota method and the Wronskian technique, respectively. Non-isospectral dynamics and source effects, including one-soliton characteristics in non-uniform media, two-solitons scattering and special behaviours related to sources (for example, the 'ghost' solitons in the degenerate two-soliton case), are investigated analytically.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Soliton equations with self-consistent sources [1–6] have received considerable attention in recent years. Physically, the sources can result in solitary waves moving with a non-constant velocity and therefore lead to a variety of dynamics of physical models. For applications, these kinds of systems are usually used to describe interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, etc [4, 5, 7]. Besides, these kinds of systems also result in many mathematically interesting treatments and recently they were investigated by means of the inverse scattering transform, Darboux transformation, bilinear method, etc [8–20].

Non-isospectral evolution equations are also of physical and mathematical importance. They are related to time-dependent spectral parameters and can describe solitary waves in non-uniform media [21, 22]. Meanwhile, the time-dependent spectral parameters will lead to generalizations [21–25] of those classical methods. In general, when the spectral parameter λ follows $\lambda_t = \lambda^j$ where $j = 0$ or 1, the corresponding non-isospectral equations are mathematically trivial since there exist transformations between them and their isospectral counterparts; while when $j > 1$, such transformation cannot be found [26]. Recently, we studied some non-isospectral evolution equations by means of the Hirota method and the Wronskian technique [27–30].

The Hirota method [31, 32] and the Wronskian technique [33] are two efficient approaches in finding exact solutions for soliton equations. Both of them are based on Hirota's bilinear form and consequently are called bilinear methods. Some soliton equations with self-consistent sources admit bilinear forms and N-soliton solutions in Hirota's expression. In addition, by means of a new determinant identity and a new verification procedure, their Wronskian solutions can also be derived [16–20].

In this paper, we aim to investigate solitons with self-consistent sources in non-uniform media, or, in other words, non-isospectral soliton equations with self-consistent sources. The non-isospectral-modified KdV equation with self-consistent sources (mKdVESCS) will be employed as an example. We first derive two non-isospectral mKdVESCSs corresponding to different time evolutions of the spectral parameter λ. One is for $\lambda_t = \lambda$ and the equation we call non-isospectral mKdVESCS-I; the other is for $\lambda_t = \lambda^3$ and we call it non-isospectral mKdVESCS-II. The non-isospectral mKdVESCS-I is mathematically trivial (but physically interesting) and there exists a gauge transformation connecting it with the isospectral mKdVESCS. The transformation is for both equations and Lax pairs and enables us to get solutions and conservation laws of the non-isospectral mKdVESCS-I from those of the isospectral mKdVESCS. Besides, the both non-isospectral mKdVESCSs can be transformed into their bilinear forms by which N-soliton solutions in Hirota's form and Wronskian's form can be obtained.

For an important part of the paper, the dynamics of solitons with sources in non-uniform media, we focus on the following three points. First, how do the characteristics of solitons, such as amplitude and velocity, rely on time and sources? Second, can the elastic scattering appear in the non-uniform media? The final one is whether sources lead to special soliton behaviours. To investigate two-soliton scattering, we employ an asymptotic analysis in the coordinate frame co-moving with a single soliton [36]. Besides, as special behaviours related to sources, the 'ghost' solitons in the degenerate two-soliton case are also described in details.

The paper is organized as follows. In section 2 the two non-isospectral mKdVESCSs are derived and their Lax pairs are given. In section 3 we list the known exact solutions to the isospectral mKdVESCS and give its infinitely many conservation laws. In section 4 we first discuss the non-isospectral mKdVESCS-I by means of gauge transformation and bilinear method (the Hirota method and the Wronskian technique), and then we investigate dynamics. Finally, in section 5 we derive solutions through bilinear method and investigate dynamics for the non-isospectral mKdVESCS-II.

2. Lax integrability of two non-isospectral mKdVESCSs

In this section we derive the non-isospectral mKdVESCS-I and II and give their Lax pairs. The method used here is essentially the same as in [17] and [18].

We start from the well-known ZS–AKNS spectral problem [34, 35]

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_x = M\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \qquad M = \begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix} \tag{2.1}$$

coupled with the time evolution

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_t = N\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \qquad N = \begin{pmatrix}A & B\\ C & -A\end{pmatrix}. \tag{2.2}$$

Their compatibility condition, i.e., the zero-curvature equation,

$$M_t - N_x + [M, N] = 0 \tag{2.3}$$

suggests that

$$A = \partial^{-1}(r, q)\begin{pmatrix}-B\\ C\end{pmatrix} - \lambda_t x + A_0 \tag{2.4}$$

$$\begin{pmatrix}q\\ r\end{pmatrix}_t = L\begin{pmatrix}-B\\ C\end{pmatrix} - 2\lambda\begin{pmatrix}-B\\ C\end{pmatrix} - 2A_0\sigma\begin{pmatrix}q\\ r\end{pmatrix} + 2\lambda_t\sigma\begin{pmatrix}xq\\ xr\end{pmatrix}, \tag{2.5}$$

where A_0 is a constant and

$$L = \sigma\partial + 2\begin{pmatrix}q\\ -r\end{pmatrix}\partial^{-1}(r, q), \tag{2.6}$$

with $\sigma = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $\partial = \frac{\partial}{\partial x}$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. To get the isospectral and non-isospectral mKdVESCSs, we take $q = -r = u$ and expand $(-B, C)^T$ as

$$\begin{pmatrix}-B\\ C\end{pmatrix} = \sum_{j=1}^{n}\begin{pmatrix}-b_j\\ c_j\end{pmatrix}(2\lambda)^{n-j} + \sum_{j=1}^{N}\left(\frac{\alpha_j}{2\lambda + 2\lambda_j} + \frac{\beta_j}{2\lambda - 2\lambda_j}\right), \tag{2.7}$$

where we let two-order column vectors α_j and β_j satisfy

$$L\alpha_j = -2\lambda_j\alpha_j, \qquad L\beta_j = 2\lambda_j\beta_j, \tag{2.8}$$

and to meet that we can take

$$\alpha_j = 2\lambda_j\begin{pmatrix}\phi_{2,j}^2\\ \phi_{1,j}^2\end{pmatrix}, \qquad \beta_j = -2\lambda_j\begin{pmatrix}\phi_{1,j}^2\\ \phi_{2,j}^2\end{pmatrix} \tag{2.9}$$

with

$$\begin{pmatrix}\phi_{1,j}\\ \phi_{2,j}\end{pmatrix}_x = \begin{pmatrix}-\lambda_j & u\\ -u & \lambda_j\end{pmatrix}\begin{pmatrix}\phi_{1,j}\\ \phi_{2,j}\end{pmatrix} \tag{2.10}$$

for $j = 1, 2, \ldots, N$. Thus (2.5) can be rewritten as

$$\begin{aligned}\begin{pmatrix}u\\ -u\end{pmatrix}_t &= \sum_{j=1}^{n} L\begin{pmatrix}-b_j\\ c_j\end{pmatrix}(2\lambda)^{n-j} - 2\lambda\sum_{j=1}^{n}\begin{pmatrix}-b_j\\ c_j\end{pmatrix}(2\lambda)^{n-j} + 2A_0\begin{pmatrix}u\\ u\end{pmatrix}\\ &\quad + 2\lambda_t\begin{pmatrix}-xu\\ -xu\end{pmatrix} - \sum_{j=1}^{N}\begin{pmatrix}\phi_{1,j}^2 + \phi_{2,j}^2\\ -\phi_{1,j}^2 - \phi_{2,j}^2\end{pmatrix}_x.\end{aligned} \tag{2.11}$$

The isospectral mKdVESCS, i.e. [8],

$$u_t + u_{xxx} + 6u^2u_x + \sum_{j=1}^{N}\left(\phi_{1,j}^2 + \phi_{2,j}^2\right)_x = 0, \tag{2.12a}$$

$$(\phi_{1,j})_x = -\lambda_j\phi_{1,j} + u\phi_{2,j}, \qquad (\phi_{2,j})_x = -u\phi_{1,j} + \lambda_j\phi_{2,j}, \qquad (j = 1, 2, \ldots, N), \tag{2.12b}$$

can be deduced from (2.11) by taking $\lambda_t = 0, n = 3, A_0 = \frac{1}{2}(2\lambda)^3, (-b_1, c_1)^T = (u, u)^T$ and $(-b_{j+1}, c_{j+1})^T = L(-b_j, c_j)^T$ for $j = 1, 2$. The corresponding A, B and C in N are described as [8]

$$A = 2\lambda u^2 + 4\lambda^3 - \sum_{j=1}^{N} \frac{2\lambda\lambda_j}{\lambda^2 - \lambda_j^2}\phi_{1,j}\phi_{2,j}, \tag{2.13a}$$

$$B = -4\lambda^2 u + 2\lambda u_x - u_{xx} - 2u^3 - \sum_{j=1}^{N}\left(\frac{\lambda_j\phi_{2,j}^2}{\lambda+\lambda_j} - \frac{\lambda_j\phi_{1,j}^2}{\lambda-\lambda_j}\right), \tag{2.13b}$$

$$C = 4\lambda^2 u + 2\lambda u_x + u_{xx} + 2u^3 - \sum_{j=1}^{N}\left(\frac{\lambda_j\phi_{2,j}^2}{\lambda-\lambda_j} - \frac{\lambda_j\phi_{1,j}^2}{\lambda+\lambda_j}\right). \tag{2.13c}$$

When $\lambda_t = -\mu\lambda$ where μ is a real constant, from (2.11) we can derive out the non-isospectral mKdVESCS-I,

$$u_t + u_{xxx} + 6u^2u_x + \mu(xu)_x + \sum_{j=1}^{N}\left(\phi_{1,j}^2 + \phi_{2,j}^2\right)_x = 0, \tag{2.14a}$$

$$(\phi_{1,j})_x = -\lambda_j\phi_{1,j} + u\phi_{2,j}, \qquad (\phi_{2,j})_x = -u\phi_{1,j} + \lambda_j\phi_{2,j}, \qquad (j = 1, 2, \ldots, N). \tag{2.14b}$$

To achieve that, we still take $n = 3, A_0 = \frac{1}{2}(2\lambda)^3, (-b_1, c_1)^T = (u, u)^T, (-b_2, c_2)^T = L(-b_1, c_1)^T$ but $(-b_3, c_3)^T = L(-b_2, c_2)^T + \mu(xu, xu)^T$. In this case, the corresponding A, B and C in N are described as

$$A = 2\lambda u^2 + 4\lambda^3 + \mu\lambda x - \sum_{j=1}^{N} \frac{2\lambda\lambda_j}{\lambda^2 - \lambda_j^2}\phi_{1,j}\phi_{2,j}, \tag{2.15a}$$

$$B = -4\lambda^2 u + 2\lambda u_x - u_{xx} - 2u^3 - \mu x v - \sum_{j=1}^{N}\left(\frac{\lambda_j\phi_{2,j}^2}{\lambda+\lambda_j} - \frac{\lambda_j\phi_{1,j}^2}{\lambda-\lambda_j}\right), \tag{2.15b}$$

$$C = 4\lambda^2 u + 2\lambda u_x + u_{xx} + 2u^3 + \mu x v - \sum_{j=1}^{N}\left(\frac{\lambda_j\phi_{2,j}^2}{\lambda-\lambda_j} - \frac{\lambda_j\phi_{1,j}^2}{\lambda+\lambda_j}\right). \tag{2.15c}$$

When $\lambda_t = -4\nu\lambda^3$, we take $n = 3, A_0 = 0, (-b_1, c_1)^T = \nu(xu, xu)^T$ and $(-b_{j+1}, c_{j+1})^T = L(-b_j, c_j)^T$ for $j = 1, 2$, then from (2.11) we can get the non-isospectral mKdVESCS-II,

$$u_t + \nu[x(u_{xxx} + 6u^2u_x) + 3u_{xx} + 4u^3 + 2u_x\partial^{-1}u^2] + \sum_{j=1}^{N}\left(\phi_{1,j}^2 + \phi_{2,j}^2\right)_x = 0, \tag{2.16a}$$

$$(\phi_{1,j})_x = -\lambda_j\phi_{1,j} + u\phi_{2,j}, \qquad (\phi_{2,j})_x = -u\phi_{1,j} + \lambda_j\phi_{2,j}, \qquad (j = 1, 2, \ldots, N). \tag{2.16b}$$

The corresponding A, B and C in N are described as

$$A = \nu[-4\lambda^3 x - 2\lambda x u^2 - 2\lambda\partial^{-1}u^2] + 2\sum_{j=1}^{N}\frac{\lambda\lambda_j}{\lambda^2-\lambda_j^2}\phi_{1,j}\phi_{2,j},$$

$$B = \nu[-4xu\lambda^2 - 2(u+xu_x)\lambda + 2u_x + xu_{xx} - 2xu^3 - 2u\partial^{-1}u^2] + \sum_{j=1}^{N}\left(\frac{\lambda_j\phi_{2,j}^2}{\lambda+\lambda_j} - \frac{\lambda_j\phi_{1,j}^2}{\lambda-\lambda_j}\right),$$

$$C = \nu[4xu\lambda^2 - 2(u+xu_x)\lambda - 2u_x + xu_{xx} + 2xu^3 + 2u\partial^{-1}u^2] + \sum_{j=1}^{N}\left(\frac{\lambda_j\phi_{2,j}^2}{\lambda-\lambda_j} - \frac{\lambda_j\phi_{1,j}^2}{\lambda+\lambda_j}\right).$$

3. Some results on the isospectral mKdVESCS

As a reference we list solutions and conservation laws of the isospectral mKdVESCS in this section.

3.1. Bilinear form and exact solutions

The mKdVESCS (2.12) can be written into the following bilinear form [16]

$$\left(D_t + D_x^3\right)\bar{f}\cdot f = 4\mathrm{i}\sum_{j=1}^{N}\bar{g}_j g_j, \tag{3.1a}$$

$$D_x^2\bar{f}\cdot f = 0, \tag{3.1b}$$

$$D_x\bar{g}_j\cdot f = -\lambda_j g_j\bar{f}, \qquad \lambda_j\in R, \qquad (j=1,2,\ldots,N) \tag{3.1c}$$

by the dependent variable transformations

$$u = \mathrm{i}\left(\ln\frac{\bar{f}}{f}\right)_x, \tag{3.2a}$$

$$\phi_{1,j} = \frac{\bar{g}_j}{\bar{f}} + \frac{g_j}{f}, \qquad \phi_{2,j} = \mathrm{i}\left(\frac{\bar{g}_j}{\bar{f}} - \frac{g_j}{f}\right), \qquad (j=1,2,\ldots,N), \tag{3.2b}$$

where $\bar{f}$ and $\bar{g}_j$ are the complex conjugates of f and g_j, and D is the well-known Hirota bilinear operator defined as [31, 32]

$$D_x^m D_t^n a\cdot b = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n a(x,t)b(x',t')|_{x'=x,t'=t}. \tag{3.3}$$

By expanding f and g_j as

$$f(x,t) = 1 + f^{(2)}\epsilon^2 + f^{(4)}\epsilon^4 + \cdots, \tag{3.4a}$$

$$g_j(x,t) = g_j^{(1)}\epsilon + g_j^{(3)}\epsilon^3 + \cdots, \qquad (j=1,2,\ldots,N). \tag{3.4b}$$

and employing the standard Hirota's procedure, the N-soliton solution to the isospectral mKdVESCS has been derived where [16]

$$f = \sum_{\mu=0,1}\exp\left\{\sum_{j=1}^{N}\mu_j\left(2\xi_j + \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\leqslant j<l\leqslant N}\mu_j\mu_l A_{jl}\right\}, \tag{3.5a}$$

$$g_h = \sqrt{\beta_h(t)}\, \mathrm{e}^{\xi_h} \sum_{\mu=0,1} \mu_h \exp\left\{ \sum_{j=1}^{h-1} \mu_j \left(2\xi_j + \frac{1}{2}A_{jh} + \frac{\pi}{2}\mathrm{i}\right) + \sum_{j=h+1}^{N} \mu_j \left(2\xi_j + \frac{1}{2}A_{jh} - \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\leqslant j<l\leqslant N, j,l\neq h} \mu_j \mu_l A_{jl} \right\},$$

$$\lambda_j = -k_j, \qquad \mathrm{e}^{A_{jl}} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2, \qquad (h = 1, 2, \ldots N), \tag{3.5b}$$

in which

$$\xi_j = k_j x - 4k_j^3 t - \int_0^t \beta_j(z)\,\mathrm{d}z + \xi_j^{(0)}, \tag{3.6}$$

$k_j, \xi_j^{(0)}$ are real constants, $k_1 < k_2 < \cdots < k_N$, $\beta_j(t)$ is an arbitrary non-negative continuous function of t defined on $(-\infty, +\infty)$, and the sum over $\mu = 0, 1$ refers to each of the $\mu_j = 0, 1$ for $j = 1, 2, \ldots, N$.

Besides Hirota's form, the N-soliton solution can also be described in terms of Wronskian.

Theorem 3.1 [16]. *The bilinear isospectral mKdVESCS (3.1) admits the following Wronskian solutions:*

$$f = |\widehat{N-1}| = W(\phi_1, \phi_2, \ldots, \phi_N), \tag{3.7a}$$

$$g_h = G_h(t)|\widehat{N-2}, \tau_h|, \qquad (h = 1, \ldots, N), \tag{3.7b}$$

where $\tau_h = (\delta_{h,1}, \delta_{h,2}, \ldots, \delta_{h,N})^T$,

$$G_h(t) = \sqrt{\beta_h(t) \prod_{l=1}^{h-1} \left(k_h^2 - k_l^2\right) \prod_{l=h+1}^{N} \left(k_l^2 - k_h^2\right)}, \tag{3.8a}$$

$$\phi_j = \mathrm{i}\,\mathrm{e}^{\xi_j} + (-1)^{j-1}\,\mathrm{e}^{-\xi_j}, \tag{3.8b}$$

ξ_j *is defined as (3.6), and we also assume that* $k_1 < k_2 < \cdots < k_N$.

To prove the theorem in [16], we made use of some properties, for example, lemma 3.1 and 3.2 in [16] and the following well-known identity [33]:

$$|M, a, b||M, c, d| - |M, a, c||M, b, d| + |M, a, d||M, b, c| = 0, \tag{3.9}$$

where M is an $N \times (N-2)$ matrix and a, b, c and d represent N column vectors.

3.2. Conservation laws

The conservation law of the AKNS isospectral evolution hierarchy equations can be described as [37]

$$(-\lambda + q\omega)_t = (A + B\omega)_x, \tag{3.10}$$

where $\omega = \frac{\varphi_2}{\varphi_1}$. $q\omega$ is determined by the Riccati equation

$$q\omega_x = -(q\omega)^2 + 2\lambda q\omega + qr \tag{3.11}$$

through the expansion

$$q\omega = \sum_{n=1}^{\infty} \frac{\omega_n}{(2\lambda)^n} \tag{3.12}$$

with

$$\omega_1 = -qr, \qquad \omega_2 = -qr_x, \tag{3.13a}$$

$$\omega_{n+1} = q\left(\frac{\omega_n}{q}\right)_x + \sum_{j=0}^{n-1} \omega_j \omega_{n-j-1} \quad (n = 2, 3, \ldots). \tag{3.13b}$$

To get the infinitely many conservation laws of the mKdVESCS, we substitute $q = -r = u$ and (2.15a), (2.15b) into (3.10) and then we have

$$(u\omega)_t = \left\{ 2\lambda u^2 - \sum_{j=1}^{N} \frac{2\lambda\lambda_j}{\lambda^2 - \lambda_j^2} \phi_{1,j}\phi_{2,j} + \left[-4\lambda^2 u + 2\lambda u_x - u_{xx} - 2u^3 - \sum_{j=1}^{N} \left(\frac{\lambda_j \phi_{2,j}^2}{\lambda + \lambda_j} - \frac{\lambda_j \phi_{1,j}^2}{\lambda - \lambda_j} \right) \right] \omega \right\}_x . \tag{3.14}$$

Then, noting that using the following formulae

$$\frac{1}{\lambda^2 - \lambda_k^2} = \frac{1}{\lambda^2} \sum_{n=0}^{\infty} \left(\frac{\lambda_k}{\lambda}\right)^{2n}, \qquad \frac{1}{\lambda \mp \lambda_k} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{\pm\lambda_k}{\lambda}\right)^{n}, \qquad \left(|\lambda| > \max_{1 \leqslant k \leqslant N} \{|\lambda_k|\}\right), \tag{3.15}$$

we compare the coefficients of same powers of λ and get infinitely many conservation laws

$$\omega_{j,t} = J_{j,x}, \qquad (j = 1, 2, \ldots), \tag{3.16}$$

where $\{\omega_j\}$ are conserved densities and $\{J_j\}$ associated fluxes.

Obviously, the mKdV equation and the mKdVESCS have same conserved densities but different associated fluxes. The first two non-trivial conserved densities are

$$\varrho_1 = \omega_1 = u^2, \tag{3.17a}$$

$$\varrho_2 = \omega_3 = uu_{xx} + u^4. \tag{3.17b}$$

In addition,

$$\varrho_0 = u \tag{3.18}$$

is also a conserved density.

4. Conservation laws and exact solutions for the non-isospectral mKdVESCS-I

4.1. Gauge transformation, exact solutions and conservation laws

There exists a gauge transformation between the non-isospectral mKdVESCS-I and the isospectral mKdVESCS. We describe this through the following theorem.

Theorem 4.1. *By the transformation*

$$S = e^{\mu t}u, \quad X = e^{-\mu t}x, \quad T = \frac{e^{-3\mu t} - 1}{-3\mu}, \quad \eta = \lambda\, e^{\mu t}, \quad \psi_k = e^{\frac{3\mu t}{2}}\phi_k, \quad (k = 1, 2), \tag{4.1a}$$

$$\eta_j = \lambda_j\, e^{\mu t}, \quad \psi_{k,j} = e^{\frac{3\mu t}{2}}\phi_{k,j}, \quad (k = 1, 2,\ j = 1, 2, \ldots, N), \tag{4.1b}$$

the non-isospectral mKdVESCS-I (2.14) can be transformed to the isospectral mKdVESCS

$$S_T + S_{XXX} + 6S^2 S_X + \left(\sum_{j=1}^{N}\psi_{1,j}^2\right)_X + \left(\sum_{j=1}^{N}\psi_{2,j}^2\right)_X = 0, \tag{4.2a}$$

$$(\psi_{1,j})_X = -\eta_j\psi_{1,j} + S\psi_{2,j}, \qquad (\psi_{2,j})_X = -S\psi_{1,j} + \eta_j\psi_{2,j} \quad (j = 1, 2, \ldots, N); \tag{4.2b}$$

and the Lax pair for the non-isospectral mKdVESCS-I given by (2.1) and (2.2) with $q = -r = u$ and (2.15) is also transformed to the Lax pair of equation (4.2), i.e.,

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_X = \begin{pmatrix}-\eta & S\\ -S & \eta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_T = \begin{pmatrix}A & B\\ C & -A\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

with

$$A = 2\eta S^2 + 4\eta^3 - \sum_{j=1}^{N}\frac{2\eta\eta_j}{\eta^2 - \eta_j^2}\psi_{1,j}\psi_{2,j},$$

$$B = -4\eta^2 S + 2\eta S_X - S_{XX} - 2S^3 - \sum_{j=1}^{N}\left(\frac{\eta_j}{\eta + \eta_j}\psi_{2,j}^2 - \frac{\eta_j}{\eta - \eta_j}\psi_{1,j}^2\right),$$

$$C = 4\eta^2 S + 2\eta S_X + S_{XX} + 2S^3 - \sum_{j=1}^{N}\left(\frac{\eta_j}{\eta - \eta_j}\psi_{2,j}^2 - \frac{\eta_j}{\eta + \eta_j}\psi_{1,j}^2\right).$$

The proof can be finished by direct verifications.

Employing the gauge transformation, we can easily get solutions and conservation laws of the non-isospectral mKdVESCS-I from the known results of the isospectral mKdVESCS (4.2).

In terms of solutions, for any $S(T, X)$ being a solution to the isospectral mKdVESCS (4.2), then

$$u(t, x) := e^{-\mu t} S\left(\frac{e^{-3\mu t} - 1}{-3\mu}, e^{-\mu t}x\right) \tag{4.4}$$

solves the non-isospectral mKdVESCS-I. For the conservation laws, if

$$\partial_T\Omega(T, X, S, S_X, \ldots, \partial_X^j S, \ldots) = \partial_X J(T, X, S, S_X, \ldots, \partial_X^j S, \ldots) \tag{4.5}$$

is a conservation law for the isospectral mKdVESCS (4.2), where Ω is a conserved density and J a flux, then

$$\begin{aligned}\partial_t&\left[e^{-\mu t}\Omega\left(\frac{e^{-3\mu t} - 1}{-3\mu}, e^{-\mu t}x, e^{\mu t}u, e^{2\mu t}u_x, \ldots, e^{(j+1)\mu t}\partial_x^j u, \ldots\right)\right]\\ &= \partial_x\left[e^{-3\mu t} J\left(\frac{e^{-3\mu t} - 1}{-3\mu}, e^{-\mu t}x, e^{\mu t}u, e^{2\mu t}u_x, \ldots, e^{(j+1)\mu t}\partial_x^j u, \ldots\right)\right.\\ &\quad \left.- \mu x\, e^{-\mu t}\Omega\left(\frac{e^{-3\mu t} - 1}{-3\mu}, e^{-\mu t}x, e^{\mu t}u, e^{2\mu t}u_x, \ldots, e^{(j+1)\mu t}\partial_x^j u, \ldots\right)\right]\end{aligned} \tag{4.6}$$

is a conservation law for the non-isospectral mKdVESCS-I (2.14). In fact, by noting that

$$\partial_X = e^{\mu t}\partial_x, \qquad \partial_T = e^{3\mu t}(\partial_t + \mu x \partial_x), \tag{4.7}$$

(4.6) can directly be derived from (4.5). Thus, if

$$Q = \int_{-\infty}^{+\infty} \Omega\big(T, X, S, S_X, \ldots, \partial_X^j S, \ldots\big)\, dX \tag{4.8}$$

is a conserved quantity for the isospectral mKdVESCS (4.2), so is

$$\widetilde{Q} = \int_{-\infty}^{+\infty} e^{-\mu t}\Omega\left(\frac{e^{-3\mu t}-1}{-3\mu}, e^{-\mu t}x, e^{\mu t}u, e^{2\mu t}u_x, \ldots, e^{(j+1)\mu t}\partial_x^j u, \ldots\right) dx \tag{4.9}$$

for the non-isospectral mKdVESCS-I (2.14). As a result, from (3.17) and (3.18), the first three non-trivial conserved densities for the non-isospectral mKdVESCS-I (2.14) are

$$\widetilde{\varrho}_0 = u, \tag{4.10a}$$

$$\widetilde{\varrho}_1 = e^{\mu t}u^2, \tag{4.10b}$$

$$\widetilde{\varrho}_2 = e^{3\mu t}uu_{xx} + e^{3\mu t}u^4. \tag{4.10c}$$

4.2. Bilinear approach

In this subsection we solve the non-isospectral mKdVESCS-I (2.14) through the Hirota method and the Wronskian technique.

By the transformation (3.2), the non-isospectral mKdVESCS-I (2.14) can be transformed into the bilinear form

$$\big(D_t + D_x^3 + \mu x D_x\big)\bar{f}\cdot f = 4i\sum_{j=1}^{N}\bar{g}_j g_j, \tag{4.11a}$$

$$D_x^2\bar{f}\cdot f = 0, \tag{4.11b}$$

$$D_x\bar{g}_j\cdot f = -\lambda_j(t)g_j\bar{f}, \qquad (j = 1, 2, \ldots, N). \tag{4.11c}$$

Then, expanding f and g as (3.4) and employing the standard Hirota's procedure, one can work out the one-soliton solution described through

$$f = 1 + i\, e^{2\theta_1}, \qquad g_1 = \sqrt{\beta_1(t)}\, e^{\theta_1}, \tag{4.12}$$

where

$$\theta_1 = c_1 e^{-\mu t}x + 4c_1^3\frac{e^{-3\mu t}-1}{3\mu} - \int_0^t \beta_1(z)\, dz + \theta_1^{(0)}, \tag{4.13}$$

c_1 and $\theta_1^{(0)}$ are arbitrary real constants, $\beta_1(t)$ is an arbitrary non-negative continuous function of t defined on $(-\infty, +\infty)$, and we have taken $\lambda_1(t) = -c_1 e^{-\mu t}$ in (4.11c).

When $N = 2$ we can get a two-soliton solution described through

$$f = 1 + i(e^{2\theta_1} + e^{2\theta_2}) - \left(\frac{c_2 - c_1}{c_2 + c_1}\right)^2 e^{2\theta_1 + 2\theta_2}, \tag{4.14a}$$

$$g_1 = \sqrt{\beta_1(t)}\left(e^{\theta_1} + i\frac{c_1 - c_2}{c_1 + c_2}e^{\theta_1 + 2\theta_2}\right), \qquad g_2 = \sqrt{\beta_2(t)}\left(e^{\theta_2} - i\frac{c_1 - c_2}{c_1 + c_2}e^{2\theta_1 + \theta_2}\right), \tag{4.14b}$$

where $\lambda_j(t) = -k_j(t) = -c_j e^{-\mu t}$ in (4.11c) and θ_j is defined as (4.13) but with subscript j instead of 1.

We can continue to work out the three-soliton solution for $N = 3$, and further for arbitrary N we can take $\lambda_j(t) = -k_j(t)$ in (4.11c) and f and g_h can still be described as the general formulae (3.5), i.e.,

$$f = \sum_{\mu=0,1} \exp\left\{\sum_{j=1}^{N} 2\mu_j\left(\theta_j + \frac{\pi}{4}\mathrm{i}\right) + \sum_{1\leqslant j<l\leqslant N} \mu_j\mu_l A_{jl}\right\}, \tag{4.15a}$$

$$g_h = \sqrt{\beta_h(t)}\,\mathrm{e}^{\theta_h} \sum_{\mu=0,1} \mu_h \exp\left\{\sum_{j=1}^{h-1} \mu_j\left(2\theta_j + \frac{1}{2}A_{jh} + \frac{\pi}{2}\mathrm{i}\right) + \sum_{j=h+1}^{N} \mu_j\left(2\theta_j + \frac{1}{2}A_{jh} - \frac{\pi}{2}\mathrm{i}\right) + \sum_{1\leqslant j<l\leqslant N, j,l\neq h} \mu_j\mu_l A_{jl}\right\}, \quad (h = 1, 2, \ldots N), \tag{4.15b}$$

where

$$k_j(t) = c_j\,\mathrm{e}^{-\mu t}, \qquad \mathrm{e}^{A_{jl}} = \left(\frac{k_j(t) - k_l(t)}{k_j(t) + k_l(t)}\right)^2 = \left(\frac{c_j - c_l}{c_j + c_l}\right)^2, \tag{4.16a}$$

$$\theta_j = c_j\,\mathrm{e}^{-\mu t}x + 4c_j^3\frac{\mathrm{e}^{-3\mu t} - 1}{3\mu} - \int_0^t \beta_j(z)\,\mathrm{d}z + \theta_j^{(0)}, \tag{4.16b}$$

with arbitrary real constants c_j and $\xi_j^{(0)}$, and the arbitrary non-negative continuous t-dependent function $\beta_j(t)$.

Besides Hirota's form, i.e., polynomials of exponential functions, the non-isospectral mKdVESCS-I (2.14) also admits solutions in Wronskian form.

Theorem 4.2. *The following Wronskians*

$$f = |\widehat{N-1}| = W(\phi_1, \phi_2, \ldots, \phi_N), \tag{4.17a}$$

$$g_h = G_h(t)|\widehat{N-2}, \tau_h|, \; (h = 1, \ldots, N) \tag{4.17b}$$

solve the bilinear non-isospectral mKdVESCS-I (4.11) where

$$G_h(t) = \sqrt{\beta_h(t)\prod_{l=1}^{h-1}\left(k_h^2(t) - k_l^2(t)\right)\prod_{l=h+1}^{N}\left(k_l^2(t) - k_h^2(t)\right)}, \tag{4.18a}$$

$$\phi_j = \mathrm{i}\,\mathrm{e}^{\theta_j} + (-1)^{j-1}\,\mathrm{e}^{-\theta_j}, \tag{4.18b}$$

$\tau_h = (\delta_{h,1}, \delta_{h,2}, \ldots, \delta_{h,N})^T$, *$k_j(t)$ and θ_j are defined as (4.16a) and (4.16b) respectively, and $\lambda_j(t) = -k_j(t)$ in (4.11c).*

Proof. By noting that each ϕ_j satisfies

$$\phi_{j,x} = -k_j(t)\bar{\phi}_j, \qquad \phi_{j,xx} = k_j^2(t)\phi_j, \qquad (j = 1, 2, \ldots, N), \tag{4.19}$$

(4.11*b*) and (4.11*c*) can be verified by taking $\lambda_j(t) = -k_j(t)$, which is the same as in [16]. So we only need to prove (4.11*a*)

By virtue of (4.19) we have

$$\bar{f} = |\bar{\phi}, \bar{\phi}^{(1)}, \ldots, \bar{\phi}^{(N-1)}| = \prod_{j=1}^{N}\frac{-1}{k_j(t)}|\phi^{(1)}, \ldots, \phi^{(N)}| = \mathcal{B}|\widetilde{N}|, \qquad \mathcal{B} = (-1)^N\prod_{j=1}^{N}\frac{1}{k_j(t)}, \tag{4.20}$$

where $\widetilde{N-j}$ indicates the set of consecutive columns $1, 2, \ldots, N-j$. Then

$$f_x = |\widehat{N-2}, N|, \qquad f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|,$$

$$f_{xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|,$$

$$\bar{f}_x = \mathcal{B}|\widetilde{N-1}, N+1|, \qquad \bar{f}_{xx} = \mathcal{B}|\widetilde{N-2}, N, N+1| + \mathcal{B}|\widetilde{N-1}, N+2|,$$

$$\bar{f}_{xxx} = \mathcal{B}|\widetilde{N-3}, N-1, N, N+1| + 2\mathcal{B}|\widetilde{N-2}, N, N+2| + \mathcal{B}|\widetilde{N-1}, N+3|.$$

Besides (4.19), ϕ_j satisfies

$$\phi_{j,t} = -4\phi_{j,xxx} - \mu x\phi_{j,x} + \beta_j(t)\bar{\phi}_j \tag{4.21}$$

for each $j = 1, 2, \ldots, N$, which leads to

$$\begin{aligned} f_t = &-4(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|) \\ &- x\mu|\widehat{N-2}, N| - \frac{(N-1)N}{2}|\widehat{N-1}| \\ &- \sum_{j=1}^{N}\beta_j(t)\sum_{l=1}^{N}(-1)^{j+l}\partial^{l-1}(\mathrm{i}\mathrm{e}^{\theta_j} - (-1)^{j-1}\,\mathrm{e}^{-\theta_j})\mathcal{A}_{j,l}, \end{aligned}$$

and

$$\begin{aligned} \bar{f}_t = &-4\mathcal{B}(|\widetilde{N-3}, N-1, N, N+1| - |\widetilde{N-2}, N, N+2| + |\widetilde{N-1}, N+3|) \\ &- \mathcal{B}x\mu|\widetilde{N-1}, N+1| - \frac{(N-1)N}{2}\mathcal{B}|\widetilde{N}| \\ &+ \sum_{j=1}^{N}\beta_j(t)\sum_{l=1}^{N}(-1)^{j+l}\partial^{l-1}(\mathrm{i}\mathrm{e}^{\theta_j} + (-1)^{j-1}\,\mathrm{e}^{-\theta_j})\bar{\mathcal{A}}_{j,l}, \end{aligned}$$

where $\mathcal{A}_{j,l}$ is the cofactor of f.

Then, with these derivatives of f in hand, by employing the same treatment for $\beta_j(t)$ for the isospectral mKdVESCS given in [16], the left-hand side of (4.11a) can be written as

$$\begin{aligned} -6\mathcal{B}(-|0, &\widetilde{N-3}, N-1, N+1||\widetilde{N-3}, N-1, N-2, N| - |\widetilde{N-3}, N-1, N, N+1| \\ &\times |0, \widetilde{N-3}, N-1, N-2| + |\widetilde{N-3}, N-1, N-2, N+1| \\ &\times |0, \widetilde{N-3}, N-1, N|) - 6\mathcal{B}(-|\widetilde{N-2}, N-1, N||0, \widetilde{N-2}, N+2| \\ &+ |\widetilde{N-2}, N-1, N+2||0, \widetilde{N-2}, N| - |\widetilde{N-2}, N, N+2||0, \widetilde{N-2}, N-1|) \\ &+ 2\mathrm{i}\sum_{j=1}^{N}(-1)^{j-1}\beta_j(t)\sum_{l=1}^{N}\sum_{h=1}^{N}(-1)^{l+h}k_j^{l+h-2}(t)[(-1)^{l-1} + (-1)^{h-1}]\bar{\mathcal{A}}_{j,l}\mathcal{A}_{j,h}. \end{aligned}$$

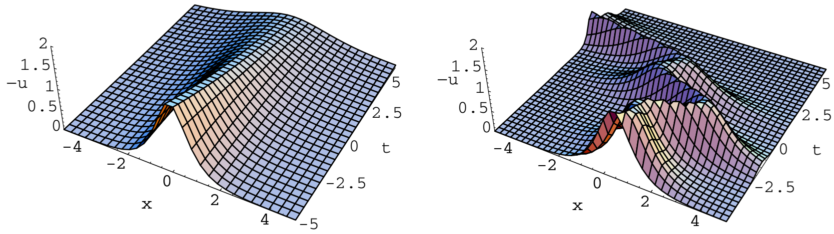
Further by means of equality (3.9), it goes to

$$2\mathrm{i}\sum_{j=1}^{N}(-1)^{j-1}\beta_j(t)\sum_{l=1}^{N}\sum_{h=1}^{N}(-1)^{l+h}k_j^{l+h-2}(t)[(-1)^{l-1} + (-1)^{h-1}]\bar{\mathcal{A}}_{j,l}\mathcal{A}_{j,h}.$$

Then, by referring to the proof for the Wronskian solution to the isospectral mKdVESCS in [16], we immediately reach the right-hand side of (4.11a) if taking g_h to be defined as (4.17b).

Thus, we complete the proof. □

Finally, we note that if setting $c_1 < c_2 < \cdots < c_N$ and employing the similar procedure as given in [16], it can be shown that the Wronskians f and g_h given by (4.17a) lead to the same solution of the non-isospectral mKdVSCS-I as the Hirota form (4.15) does.



(*a*) (*b*)

Figure 1. Shape and motion of one-soliton of the non-isospectral mKdVESCS-I. (*a*) A stationary soliton $(-u)$ given by (4.22) for $\mu = 0.1, c_1 = -0.5, \beta_1(t) = -4c_1^3\,\mathrm{e}^{-3\mu t}$ and $\theta_1^{(0)} = 0$. (*b*) A moving soliton given by (4.22) for $\mu = 0.1, c_1 = -0.5, \beta_1(t) = 1 - \sin 2t$ and $\theta_1^{(0)} = 0$.

4.3. Dynamics

Now we consider one-soliton characteristics and two-soliton scattering.

4.3.1. One-soliton characteristics. From (3.2) and (4.12) we have the one-soliton

$$u = 2c_1\,\mathrm{e}^{-\mu t}\operatorname{sech} 2\theta_1, \tag{4.22}$$

and the corresponding sources

$$\phi_{1,1} = \sqrt{\beta_1(t)}\,\mathrm{e}^{-\theta_1}\operatorname{sech} 2\theta_1, \qquad \phi_{2,1} = -\sqrt{\beta_1(t)}\,\mathrm{e}^{\theta_1}\operatorname{sech} 2\theta_1, \tag{4.23}$$

where θ_1 is given by (4.13).

(4.22) provides a soliton travelling with a time-dependent amplitude $2|c_1|\,\mathrm{e}^{-\mu t}$ and top trace

$$x(t) = \mathrm{e}^{\mu t}\int_0^t \left(\frac{\beta_1(z)}{c_1} + 4c_1^2\,\mathrm{e}^{-3\mu z}\right)\mathrm{d}z - \frac{\theta_1^{(0)}}{c_1}\,\mathrm{e}^{\mu t}, \tag{4.24}$$

or velocity

$$x'(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{e}^{\mu t}}{3c_1}\left(4c_1^3 + 8c_1^3\,\mathrm{e}^{-3\mu t} - 3\mu\theta_1^{(0)} + 3\beta_1(t) + 3\mu\int_0^t \beta_1(z)\,\mathrm{d}z\right). \tag{4.25}$$

Noting that $\int_{-\infty}^{+\infty} u\,\mathrm{d}x$ is a conserved quantity of the non-isospectral mKdVESCS-I, although when $\mu > 0$ the amplitude of (4.22) decreases as $t \to +\infty$, the soliton u is not damped and the wave becomes wider and wider. The non-negative function $\beta_1(t)$ plays the role of source and it changes the velocity of the soliton but not the shape. We can have a variety of travelling trajectories by choosing different $\beta_1(t)$. One such special case is that when $c_1 < 0, \mu\theta_1^{(0)} \geqslant 0$ and $\beta_1(t) = -4c_1^3\,\mathrm{e}^{-3\mu t} + \mu\theta_1^{(0)}\,\mathrm{e}^{-\mu t}$, (4.25) turns out to be zero, and we will have a stationary soliton with the top line $x \equiv -\frac{\theta_1^{(0)}}{c_1}$, as described in figure 1(*a*).

4.3.2. Two-soliton scattering. When f, g_1 and g_2 are defined by (4.14), (3.2*a*) provides the two-soliton solution and describes interactions between two solitons. Let us call these two solitons θ_1-soliton and θ_2-soliton for convenience. (3.2*b*) provides the corresponding sources.

Although the non-isospectral mKdVESCS-I describes solitons in non-uniform media, its solitons can scatter elastically under some conditions, as shown in figure 2(*a*).

In what follows we set $|c_2| > |c_1| > 0$ and discuss the two-soliton scattering in detail. We first suppose that $c_2 > c_1 > 0$ and

$$\int_0^t \left[\frac{\beta_1(t)}{c_1} - \frac{\beta_2(t)}{c_2} + 4\left(c_1^2 - c_2^2\right)\mathrm{e}^{-3\mu t}\right]\mathrm{d}t \to \mp\infty, \qquad \text{as} \quad t \to \pm\infty. \tag{4.26}$$

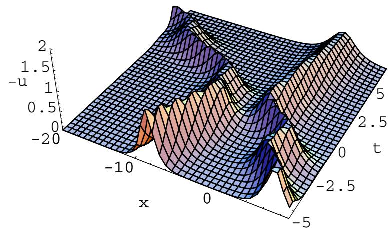


Figure 2. Two-soliton scattering of the non-isospectral mKdVESCS-I. The shape and motion of the two-soliton solution $(-u)$ given by (3.2) and (4.14a) for $\mu = 0.06, c_1 = -0.5, c_2 = -0.4, \beta_1(t) = 0.15, \beta_1(t) = 1 - \sin 2t$ and $\theta_1^{(0)} = \theta_2^{(0)} = 0$.

We also straighten the travelling trajectories of θ_j-soliton and introduce

$$X = \mathrm{e}^{-\mu t}x - \int_0^t \left(\frac{\beta_1(t)}{c_1} + 4c_1^2\,\mathrm{e}^{-3\mu t}\right)\mathrm{d}t + \frac{\theta_1^{(0)}}{c_1}, \tag{4.27}$$

$$Y = \mathrm{e}^{-\mu t}x - \int_0^t \left(\frac{\beta_2(t)}{c_2} + 4c_2^2\,\mathrm{e}^{-3\mu t}\right)\mathrm{d}t + \frac{\theta_2^{(0)}}{c_2}. \tag{4.28}$$

Then, we consider these two solitons in the coordinate frame (X, t). If the frame co-moves with θ_1-soliton, i.e., $\theta_1 = c_1 X$ stays constant, we have

$$\theta_2 = c_2 X + c_2 \int_0^t \left[\frac{\beta_1(t)}{c_1} - \frac{\beta_2(t)}{c_2} + 4\left(c_1^2 - c_2^2\right)\mathrm{e}^{-3\mu t}\right]\mathrm{d}t + \theta_2^{(0)} - \frac{c_2}{c_1}\theta_1^{(0)} \to \mp\infty, \qquad \text{as} \quad t \to \pm\infty, \tag{4.29}$$

provided (4.26) is satisfied. That suggests

$$u = i\left(\ln\frac{\bar f}{f}\right)_X \cdot \frac{\partial X}{\partial x} \to \begin{cases} 2c_1\,\mathrm{e}^{-\mu t}\,\mathrm{sech}\,2\theta_1, & t \to +\infty, \\ 2c_1\,\mathrm{e}^{-\mu t}\,\mathrm{sech}\,2\left(\theta_1 + \ln\dfrac{c_2 - c_1}{c_2 + c_1}\right), & t \to -\infty. \end{cases}$$

Thus we have extracted out the initial and final states of θ_1-soliton, and it then follows that θ_1-soliton gets a leftward phase shift $-\frac{1}{c_1}\ln\frac{c_2-c_1}{c_2+c_1}$ with respect to the frame (X, t) after interaction. Similarly, we consider the two solitons in the coordinate frame (Y, t) co-moving with θ_2-soliton where we let θ_2 stay constant and $t \to \pm\infty$. We have

$$\theta_1 = c_1 Y - c_1 \int_0^t \left[\frac{\beta_1(t)}{c_1} - \frac{\beta_2(t)}{c_2} + 4\left(c_1^2 - c_2^2\right)\mathrm{e}^{-3\mu t}\right]\mathrm{d}t + \theta_1^{(0)} - \frac{c_1}{c_2}\theta_2^{(0)} \to \pm\infty, \quad \text{as} \quad t \to \pm\infty, \tag{4.30}$$

still under condition (4.26), and further

$$u = i\left(\ln\frac{\bar f}{f}\right)_Y \cdot \frac{\partial Y}{\partial x} \to \begin{cases} 2c_2\,\mathrm{e}^{-\mu t}\,\mathrm{sech}\,2\left(\theta_2 + \ln\dfrac{c_2 - c_1}{c_2 + c_1}\right), & t \to +\infty, \\ 2c_2\,\mathrm{e}^{-\mu t}\,\mathrm{sech}\,2\theta_2, & t \to -\infty. \end{cases}$$

That means θ_2-soliton gets a rightward phase shift $-\frac{1}{c_2}\ln\frac{c_2-c_1}{c_2+c_1}$ with respect to the frame (Y, t) after interaction.

After similar discussions for other cases of $|c_2| > |c_1| > 0$, we reach the following result.

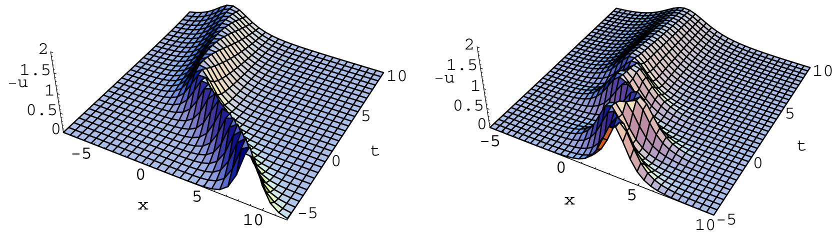


Figure 3. The degenerate case ($c_1 = c_2$) of two-soliton interactions of the non-isospectral mKdVESCS-I. (*a*) The shape and motion of the solution ($-u$) given by (3.2*a*) and (4.32) with $\mu = 0.05, c_1 = c_2 = -0.5, \beta_1(t) = 0.1 - 4c_1^3\,\mathrm{e}^{-3\mu t}, \beta_2(t) = 1 - 4c_1^3\,\mathrm{e}^{-3\mu t}$ and $\theta_1^{(0)} = \theta_2^{(0)} = 0$. (*b*) Thw shape and motion of the solution ($-u$) given by (3.2*a*) and (4.32) with $\mu = 0.05, c_1 = c_2 = -0.5, \beta_1(t) = -4c_1^3\,\mathrm{e}^{-3\mu t}, \beta_2(t) = 1 - \sin 2t$ and $\theta_1^{(0)} = \theta_2^{(0)} = 0$.

Theorem 4.3. *There will be scattering between two solitons with sources defined by (3.2a) and (4.14a) when they satisfy (4.26) and* $|c_2| > |c_1| > 0$. *After scattering* θ_1*-soliton gets a leftward phase shift* $\left|\frac{1}{c_1}\ln\frac{c_2-c_1}{c_2+c_1}\right|$ *in the frame* (X, t) *and* θ_2*-soliton gets a rightward phase shift* $\left|\frac{1}{c_2}\ln\frac{c_2-c_1}{c_2+c_1}\right|$ *in the frame* (Y, t). *If, instead of (4.26),*

$$\int_0^t \left[\frac{\beta_1(t)}{c_1} - \frac{\beta_2(t)}{c_2} + 4\left(c_1^2 - c_2^2\right)\mathrm{e}^{-3\mu t}\right]\mathrm{d}t \to \pm\infty, \qquad \text{as} \quad t \to \pm\infty, (|c_2| > |c_1| > 0), \tag{4.31}$$

then the soliton scattering also exists while after scattering θ_1*-soliton gets a rightward phase shift* $\left|\frac{1}{c_1}\ln\frac{c_2-c_1}{c_2+c_1}\right|$ *in the frame* (X, t) *and* θ_2*-soliton gets a leftward phase shift* $\left|\frac{1}{c_2}\ln\frac{c_2-c_1}{c_2+c_1}\right|$ *in the frame* (Y, t).

In the following we consider the degenerate case ($c_1 = c_2$) of two-soliton interactions. In this case,

$$f = 1 + \mathrm{i}(\mathrm{e}^{2\theta_1} + \mathrm{e}^{2\theta_2}), \tag{4.32}$$

$$g_j = \sqrt{\beta_j(t)}\,\mathrm{e}^{\theta_j}, \qquad (j = 1, 2), \tag{4.33}$$

where

$$\theta_j = c_1\,\mathrm{e}^{-\mu t}x + 4c_1^3\frac{\mathrm{e}^{-3\mu t} - 1}{3\mu} - \int_0^t \beta_j(z)\,\mathrm{d}z + \theta_j^{(0)}, \qquad (j = 1, 2). \tag{4.34}$$

This seems essentially to be an one-soliton with a special source, but we would like to look at it as a degenerate two-soliton case for it has two different source representatives g_1 and g_2.

Figure 3(*a*) and (*b*) describe the special behaviours of such degenerate two-soliton solutions, respectively, where the two solitons travel first with their original sources and then suddenly with other different sources. Corresponding to the 'ghost' solitons of the Hirota–Satsuma equation [38, 36], in our case, the soliton u also shows 'ghost' behaviours.

Let us give more details in the following. Suppose that $\beta_j(t)$ satisfy

$$\int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t \to \mp\infty, \qquad \text{as} \quad t \to \pm\infty \tag{4.35}$$

and we consider the following coordinate frame co-moving with θ_1-soliton,

$$\left(X = \mathrm{e}^{-\mu t}x - \int_0^t \left(\frac{\beta_1(t)}{c_1} + 4c_1^2\,\mathrm{e}^{-3\mu t}\right)\mathrm{d}t + \frac{\theta_1^{(0)}}{c_1}, t\right), \tag{4.36}$$

where θ_1 stays constant but, due to (4.35),

$$\theta_2 = c_1 X + \int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t + \theta_2^{(0)} - \theta_1^{(0)} \to \mp\infty, \qquad \text{as} \quad t \to \pm\infty. \tag{4.37}$$

It then follows that

$$u \to \begin{cases} 2c_1\,\mathrm{e}^{-\mu t}\mathrm{sech}(2\theta_1), & t \to +\infty, \\ 0, & t \to -\infty, \end{cases}$$

which means θ_1-soliton does not exist initially but appears finally. Similarly, under the frame

$$\left(Y = \mathrm{e}^{-\mu t} x - \int_0^t \left(\frac{\beta_2(t)}{c_1} + 4c_1^2\,\mathrm{e}^{-3\mu t}\right)\mathrm{d}t + \frac{\theta_2^{(0)}}{c_1}, t\right) \tag{4.38}$$

which co-moves with θ_2-soliton, we have

$$u \to \begin{cases} 0, & t \to +\infty, \\ 2c_1\,\mathrm{e}^{-\mu t}\mathrm{sech}(2\theta_2), & t \to -\infty, \end{cases}$$

which means θ_2-soliton exists initially but disappears finally.

Similar results can be obtained for $\beta_j(t)$ satisfying

$$\int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t \to \pm\infty, \qquad \text{as} \quad t \to \pm\infty. \tag{4.39}$$

Thus, we have shown how the sources play roles in the degenerate two-soliton case.

5. The non-isospectral mKdVESCS-II

In this section, we first derive exact solutions to the non-isospectral mKdVESCS-II via the Hirota method and the Wronskian technique. Then we will investigate dynamics of the solutions obtained.

5.1. Bilinear form and the Hirota method

Let us first transform the non-isospectral mKdVESCS-II (2.16) into its following bilinear form:

$$\left(D_t + x\nu D_x^3\right)\bar{f}\cdot f + 2\nu(f\bar{f}_{xx} - \bar{f}f_{xx}) = 4\mathrm{i}\sum_{j=1}^{N}\bar{g}_j g_j, \tag{5.1a}$$

$$D_x^2\bar{f}\cdot f = 0, \tag{5.1b}$$

$$D_x\bar{g}_j\cdot f = -\lambda_j(t) g_j\bar{f}, \quad (j = 1, 2, \ldots, N), \tag{5.1c}$$

where the transformation we used is still (3.2).

Expanding f and g_h as (3.4), we can derive multi-soliton solutions following Hirota's approach. For $N = 1$, one soliton can be described through

$$f = 1 + i\,\mathrm{e}^{2\widetilde{\zeta}_1}, \qquad g_1 = \sqrt{\beta_1(t)}\,\mathrm{e}^{\widetilde{\zeta}_1}, \tag{5.2}$$

where

$$\widetilde{\zeta}_1 = k_1(t)x - \int_0^t \left(\beta_1(z) + \frac{4\nu}{8\nu z + c_1}\right)\mathrm{d}z + \widetilde{\zeta}_1^{(0)}, \qquad k_1(t) = \frac{1}{\sqrt{8\nu t + c_1}}, \qquad \nu t > -\frac{c_1}{8}, \tag{5.3}$$

with arbitrary real constants $c_1 > 0$ and $\widetilde{\zeta}_1^{(0)}$ and the arbitrary non-negative continuous time-dependent function $\beta_1(t)$; and we have taken $\lambda_1(t) = -k_1(t)$ in (5.1*c*).

For $N = 2$, we can obtain the two-soliton solution which is described through

$$f = 1 + \mathrm{i}(\mathrm{e}^{2\widetilde{\zeta}_1} + \mathrm{e}^{2\widetilde{\zeta}_2}) - \left(\frac{k_1(t) - k_2(t)}{k_1(t) + k_2(t)}\right)^2 \mathrm{e}^{2\widetilde{\zeta}_1 + 2\widetilde{\zeta}_2}, \tag{5.4a}$$

$$g_1 = \sqrt{\beta_1(t)}\,\mathrm{e}^{\widetilde{\zeta}_1}\left(1 + \mathrm{i}\frac{k_1(t) - k_2(t)}{k_1(t) + k_2(t)}\,\mathrm{e}^{2\widetilde{\zeta}_2}\right), \qquad g_2 = \sqrt{\beta_2(t)}\,\mathrm{e}^{\widetilde{\zeta}_2}\left(1 - \mathrm{i}\frac{k_1(t) - k_2(t)}{k_1(t) + k_2(t)}\,\mathrm{e}^{2\widetilde{\zeta}_1}\right). \tag{5.4b}$$

where we take $\lambda_j(t) = -k_j(t)$ in (5.1*c*) and

$$\widetilde{\zeta}_j = k_j(t)x - \int_0^t \left(\beta_j(z) + \frac{4\nu}{8\nu z + c_j}\right)\mathrm{d}z + \widetilde{\zeta}_j^{(0)}, \qquad k_j(t) = \frac{1}{\sqrt{8\nu t + c_j}}, \tag{5.5}$$

with arbitrary real constants $c_j > 0$ and $\widetilde{\zeta}_j^{(0)}$ and the arbitrary non-negative continuous time-dependent function $\beta_j(t)$.

For arbitrary N, the N-soliton solution can be obtained by taking $\lambda_j(t) = -k_j(t)$ in (5.1*c*) and

$$f = \sum_{\mu=0,1} \exp\left\{\sum_{j=1}^{N} 2\mu_j\left(\widetilde{\zeta}_j + \frac{\pi \mathrm{i}}{4}\right) + \sum_{1\leqslant j<l\leqslant N} \mu_j\mu_l A_{jl}\right\}, \tag{5.6a}$$

$$\begin{aligned} g_h = \sqrt{\beta_h(t)}\,\mathrm{e}^{\widetilde{\zeta}_h} \sum_{\mu=0,1} \mu_h \exp\Bigg\{&\sum_{j=1}^{h-1}\mu_j\left[2\widetilde{\zeta}_j + \frac{1}{2}A_{jh} + \frac{\pi}{2}\mathrm{i}\right] + \sum_{j=h+1}^{N}\mu_j\left[2\widetilde{\zeta}_j + \frac{1}{2}A_{jh} - \frac{\pi}{2}\mathrm{i}\right] \\ &+ \sum_{1\leqslant j<l\leqslant N, j,l\neq h} \mu_j\mu_l A_{jl}\Bigg\}, \qquad (h = 1, 2, \ldots N), \end{aligned} \tag{5.6b}$$

where $\widetilde{\zeta}_j$ is defined as (5.5),

$$\mathrm{e}^{A_{jl}} = \left(\frac{k_j(t) - k_l(t)}{k_j(t) + k_l(t)}\right)^2 = \left(\frac{\frac{1}{\sqrt{8\nu t + c_j}} - \frac{1}{\sqrt{8\nu t + c_l}}}{\frac{1}{\sqrt{8\nu t + c_j}} + \frac{1}{\sqrt{8\nu t + c_l}}}\right)^2, \tag{5.7}$$

and we set $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N > 0$ without loss of generality and also set $\nu t > -\frac{c_N}{8}$ to avoid singularities.

5.2. Solutions in the Wronskian form

Theorem 5.1. *The bilinear non-isospectral mKdVESCS-II (5.1) admits the following Wronskian solutions:*

$$f = |\widehat{N-1}| = W(\phi_1, \phi_2, \ldots, \phi_N), \tag{5.8a}$$

$$g_h = G_h(t)|\widehat{N-2}, \tau_h|, \qquad (h = 1, \ldots, N) \tag{5.8b}$$

where

$$G_h(t) = \sqrt{\beta_h(t)\prod_{l=1}^{h-1}\left(k_h^2(t) - k_l^2(t)\right)\prod_{l=h+1}^{N}\left(k_l^2(t) - k_h^2(t)\right)}, \tag{5.9a}$$

$$\phi_j = \mathrm{i}\,\mathrm{e}^{\zeta_j} + (-1)^{j-1}\,\mathrm{e}^{-\zeta_j}, \tag{5.9b}$$

$\tau_h = (\delta_{h,1}, \delta_{h,2}, \ldots, \delta_{h,N})^T$, $\lambda_j(t) = -k_j(t)$ in (5.1c) and ζ_j are defined as

$$\zeta_j = k_j(t)x - \int_0^t \beta_j(z)\,\mathrm{d}z + \zeta_j^{(0)}, \qquad k_j(t) = \frac{1}{\sqrt{8\nu t + c_j}} \tag{5.10}$$

with arbitrary real constants $c_j > 0$ and $\zeta_j^{(0)}$ and the arbitrary non-negative continuous time-dependent function $\beta_j(t)$.

To achieve the proof, one should notice that ϕ_j satisfies

$$\phi_{j,t} = -4x\nu\phi_{j,xxx} + \beta_j(t)\bar{\phi}_j. \tag{5.11}$$

which implies

$$\partial_t\left(\partial_x^l\phi_j\right) \equiv \phi_{j,t}^{(l)} = -4x\nu\phi_j^{(l+3)} - 4l\nu\phi_j^{(l+2)} + \beta_j(t)\bar{\phi}_j^{(l)}$$

for $l = 0, 1, \ldots$, and this leads to

$$\begin{aligned} f_t = {} & -4x\nu(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|) \\ & + \nu[4(N-2)|\widehat{N-3}, N-1, N| - 4(N-1)|\widehat{N-2}, N+1|] \\ & - \sum_{j=1}^{N}\beta_j(t)\sum_{l=1}^{N}(-1)^{j+l}\partial^{l-1}(\mathrm{i}\mathrm{e}^{\theta_j} - (-1)^{j-1}\,\mathrm{e}^{-\theta_j})\mathcal{A}_{j,l}, \end{aligned}$$

$$\begin{aligned} \bar{f}_t = {} & -4\nu\mathcal{B}x(|\widetilde{N-3}, N-1, N, N+1| - |\widetilde{N-2}, N, N+2| + |\widetilde{N-1}, N+3|) \\ & - 4\nu\mathcal{B}[-(N-2)|\widetilde{N-2}, N, N+1| + (N-1)|\widetilde{N-1}, N+2|] \\ & + \sum_{j=1}^{N}\beta_j(t)\sum_{l=1}^{N}(-1)^{j+l}\partial^{l-1}(\mathrm{i}\mathrm{e}^{\theta_j} + (-1)^{j-1}\,\mathrm{e}^{-\theta_j})\bar{\mathcal{A}}_{j,l}, \end{aligned}$$

where $\mathcal{A}_{j,l}$ is the cofactor of f and $\mathcal{B} = (-1)^N\prod_{j=1}^{N}\frac{1}{k_j(t)}$. Then the rest part of the proof is similar to the bilinear non-isospectral mKdV-I, so we skip the details.

Before we discuss dynamics, we note that for the non-isospectral mKdVESCS-II we cannot uniform the solutions in Hirota's form and Wronskian's form, as done in [16] for the isospectral mKdVESCS. In fact, following the procedure in [16] and setting $c_1 > c_2 > \cdots > c_N > 0$, we can rewrite (5.8a) and (5.8b) as

$$f = F(t)\sum_{\mu=0,1}\exp\left\{\sum_{j=1}^{N}2\mu_j\left(\eta_j + \frac{\pi\mathrm{i}}{4}\right) + \sum_{1\leqslant j<l\leqslant N}\mu_j\mu_l A_{jl}\right\}, \tag{5.12a}$$

$$\begin{aligned} g_h = F(t)\sqrt{\beta_h(t)}\,\mathrm{e}^{\eta_h}\sum_{\mu=0,1}\mu_h\exp & \left\{\sum_{j=1}^{h-1}\mu_j\left(2\eta_j + \frac{A_{jh}}{2} + \frac{\pi\mathrm{i}}{2}\right) + \sum_{j=h+1}^{N}\mu_j\left(2\eta_j + \frac{A_{jh}}{2} - \frac{\pi\mathrm{i}}{2}\right)\right. \\ & \left. + \sum_{1\leqslant j<l\leqslant N, j,l\neq h}\mu_j\mu_l A_{jl}\right\}, \qquad (h = 1, 2, \ldots N), \end{aligned} \tag{5.12b}$$

where

$$\eta_j = \zeta_j - \frac{1}{4}\sum_{l=1, l\neq j}^{N} A_{jl}, \qquad F(t) = \left(\prod_{j=1}^{N} \mathrm{e}^{-\zeta_j}\right)\left[\prod_{1\leqslant j<l\leqslant N,} (k_l(t) - k_j(t))\right], \tag{5.13}$$

and $\mathrm{e}^{A_{jl}}$ is defined as (5.7).

5.3. Dynamics

5.3.1. One-soliton characteristics. When $N = 1$ from (5.8) we have

$$f = \mathrm{i}\,\mathrm{e}^{\zeta_1} + \mathrm{e}^{-\zeta_1}, \qquad g_1 = \sqrt{\beta_1(t)}. \tag{5.14}$$

It then follows from (3.2) that

$$u = \frac{2}{\sqrt{8\nu t + c_1}}\,\mathrm{sech}\, 2\zeta_1, \tag{5.15}$$

$$\phi_{1,1} = \sqrt{\beta_1(t)}\,\mathrm{e}^{-\zeta_1}\,\mathrm{sech}\, 2\zeta_1, \qquad \phi_{2,1} = -\sqrt{\beta_1(t)}\,\mathrm{e}^{\zeta_1}\,\mathrm{sech}\, 2\zeta_1, \tag{5.16}$$

where ζ_1 is given by (5.10).

If we start from (5.2), we have

$$u = \frac{2}{\sqrt{8\nu t + c_1}}\,\mathrm{sech}\, 2\widetilde{\zeta}_1, \tag{5.17}$$

$$\phi_{1,1} = \sqrt{\beta_1(t)}\,\mathrm{e}^{-\widetilde{\zeta}_1}\,\mathrm{sech}\, 2\widetilde{\zeta}_1, \qquad \phi_{2,1} = -\sqrt{\beta_1(t)}\mathrm{e}^{\widetilde{\zeta}_1}\,\mathrm{sech}\, 2\widetilde{\zeta}_1, \tag{5.18}$$

where $\widetilde{\zeta}_1$ is given by (5.5). Obviously, when $\nu > 0$ all the u given by (5.17) can also be obtained from (5.15). So, we would like to consider the Wronskian solution for one soliton in this part.

(5.15) provides a soliton travelling with a decay amplitude $\frac{2}{\sqrt{8\nu t + c_1}}$ (when $\nu > 0$) and the time-dependent top trace

$$x(t) = \sqrt{8\nu t + c_1}\left(\int_0^t \beta_1(z)\,\mathrm{d}z - \zeta_1^{(0)}\right), \tag{5.19}$$

or velocity

$$x'(t) = \frac{(8\nu t + c_1)\beta_1(t) + 4\nu\left(\int_0^t \beta_1(z)\,\mathrm{d}z - \zeta_1^{(0)}\right)}{\sqrt{8\nu t + c_1}}. \tag{5.20}$$

The non-negative function $\beta_1(t)$ plays the role of source and it changes the velocity of the soliton but not the shape. The stationary soliton can be obtained when we take $\nu\zeta_1^{(0)} \geqslant 0$ and $\beta_1(t) = 4\nu\zeta_1^{(0)}\sqrt{c_1}(8\nu t + c_1)^{-\frac{3}{2}}$. In this case, $x(t) \equiv -\zeta_1^{(0)}\sqrt{c_1}$ and $x'(t) \equiv 0$. Figure 4 describes the shape and motion of one soliton.

5.3.2. Two-soliton scattering. Because the two-soliton solution in the Wronskian form cannot provide a non-trivial degenerate solution, we consider the two-soliton solution in Hirota's form in this part, i.e., f, g_1 and g_2 are given by (5.4).

We first suppose that $c_1 > c_2 > 0$ and $\nu t > -\frac{c_2}{8}$. In this case, it is not easy to analytically investigate two-soliton interactions, but they do have elastic scattering. Figure 5(*a*) exhibits the elastic interactions of two solitons with decay amplitudes. For comparison we give the two corresponding single solitons in figure 5(*b*) and (*c*).

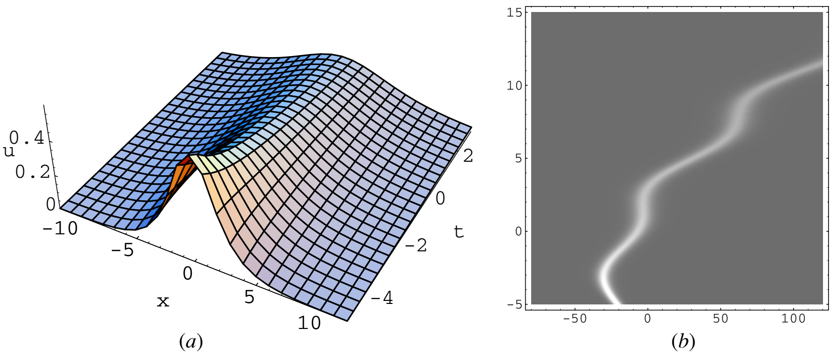


Figure 4. The shape and motion of one soliton of the non-isospectral mKdVESCS-II. (*a*) A stationary soliton given by (5.15) for $\nu = 1, c_1 = 50, \beta_1(t) = 4\nu\zeta_1^{(0)}\sqrt{c_1}(8\nu t+c_1)^{-\frac{3}{2}}$ and $\zeta_1^{(0)} = 0$. (*b*) The density plot of a moving soliton given by (5.15) for $\nu = 1, c_1 = 50, \beta_1(t) = 1 - \sin t$ and $\zeta_1^{(0)} = 1, x \in [-50, 120], t \in [-5, 15]$ and plot range $[-0.3, 0.4]$. The grey area denotes zero value and bright strap denotes positive soliton.

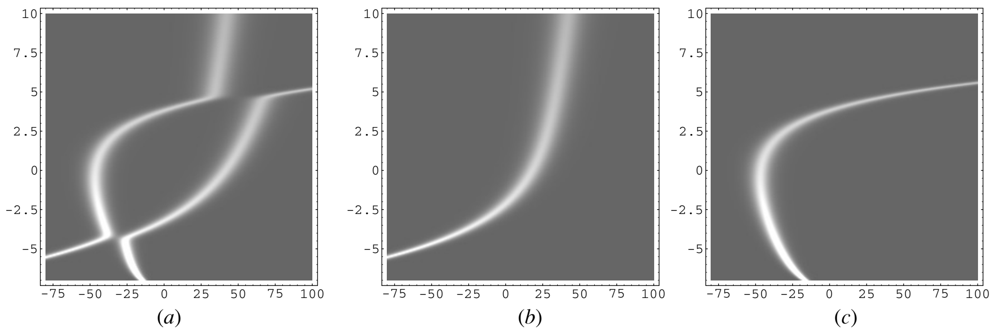


Figure 5. One-soliton behaviours and two-soliton interactions of the non-isospectral mKdVESCS-II. (*a*) The density plot of the two-soliton solution given by (3.2) and (5.4) for $\nu = 1, c_1 = 80, c_2 = 60, \beta_1(t) = 0.5\,\mathrm{e}^{-0.5t}, \beta_2(t) = 0.5\,\mathrm{e}^{0.5t}, \zeta_1^{(0)} = -2, \zeta_2^{(0)} = 6, x \in [-80, 100], t \in [-7, 10]$ and plot range $[-0.2, 0.3]$. (*b*) The density plot of the corresponding one soliton given by (5.17) for $\nu = 1, c_1 = 80, \beta_1(t) = 0.5\,\mathrm{e}^{-0.5t}, \zeta_1^{(0)} = -2, x \in [-80, 100], t \in [-7, 10]$ and plot range $[-0.2, 0.3]$. (*c*) The density plot of the corresponding one soliton given by (5.17) for $\nu = 1, c_1 = 60, \beta_1(t) = 0.5\,\mathrm{e}^{0.5t}, \zeta_1^{(0)} = 6, x \in [-80, 100], t \in [-7, 10]$ and plot range $[-0.2, 0.3]$. The grey area denotes zero value and bright straps denote positive solitons.

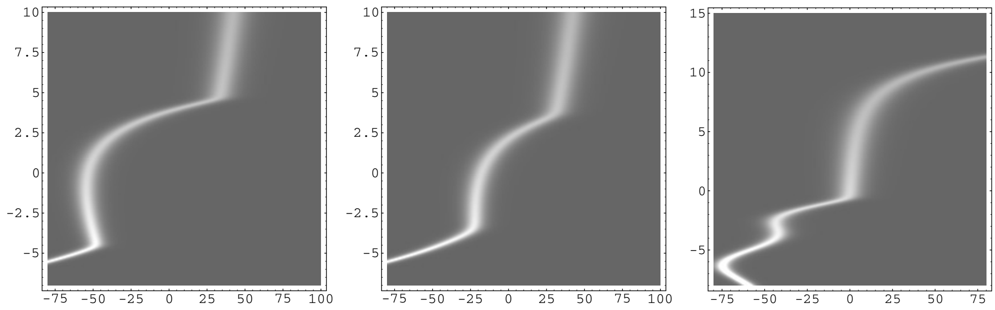
In the following let us consider the degenerate two-soliton solutions. When $c_1 = c_2 > 0$ in (5.4) we have

$$f = 1 + \mathrm{i}(\mathrm{e}^{2\widetilde{\zeta}_1} + \mathrm{e}^{2\widetilde{\zeta}_2}), \qquad g_1 = \sqrt{\beta_1(t)}\,\mathrm{e}^{\widetilde{\zeta}_1}, \qquad g_2 = \sqrt{\beta_2(t)}\,\mathrm{e}^{\widetilde{\zeta}_2}, \tag{5.21}$$

where

$$\widetilde{\zeta}_j = \frac{x}{\sqrt{8\nu t + c_1}} - \int_0^t \left(\beta_j(z) + \frac{4\nu}{8\nu z + c_1}\right) \mathrm{d}z + \widetilde{\zeta}_j^{(0)}, \qquad j = 1, 2.$$

Figure 6(*a*) and (*b*) describe that a single soliton is 'disturbed' by an invisible 'ghost' soliton. In (*c*), a soliton travels first with its original source and then suddenly with another different source.



(*a*) (*b*) (*c*)

Figure 6. The degenerate two-soliton solution ($c_1 = c_2$) of the non-isospectral mKdVESCS-II. (*a*) The density plot of the degenerate two-soliton solution given by (3.2) and (5.21) for $\nu = 1$, $c_1 = c_2 = 80$, $\beta_1(t) = 0.5\,\mathrm{e}^{-0.5t}$, $\beta_2(t) = 0.5\,\mathrm{e}^{0.5t}$, $\widetilde{\zeta}_1^{(0)} = -2$, $\widetilde{\zeta}_2^{(0)} = 6$, $x \in [-80, 100]$, $t \in [-7, 10]$ and plot range $[-0.2, 0.3]$. (*b*) The density plot of the degenerate two-soliton solution given by (3.2) and (5.21) with same parameters as (*a*) except $\widetilde{\zeta}_2^{(0)} = 2$ instead of 6. Thus one can see the existence of $\widetilde{\zeta}_2$-soliton by comparing (*a*) and (*b*). (*c*) The density plot of the degenerate two-soliton solution given by (3.2) and (5.21) for $\nu = 1$, $c_1 = c_2 = 80$, $\beta_1(t) = 0.01\,\mathrm{e}^{0.5t}$, $\beta_2(t) = 2(1 - \sin 1.5t)$, $\widetilde{\zeta}_1^{(0)} = 0$, $\widetilde{\zeta}_2^{(0)} = -2$, $x \in [-80, 80]$, $t \in [-8, 15]$ and plot range $[-0.2, 0.3]$. The grey area denotes zero value and bright straps denote positive solitons.

To obtain more details for such degenerate case we investigate the asymptotic behaviours. Suppose that $\beta_j(t)$ satisfy

$$\int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t \to -\infty, \qquad \text{as} \quad t \to +\infty \tag{5.22}$$

and the two solitons involved in the degenerate case are called $\widetilde{\zeta}_1$-soliton and $\widetilde{\zeta}_2$-soliton, respectively. We consider the coordinate frame co-moving with $\widetilde{\zeta}_1$-soliton,

$$\left(X = \widetilde{\zeta}_1 = \frac{x}{\sqrt{8\nu t + c_1}} - \int_0^t \left(\beta_1(z) + \frac{4\nu}{8\nu z + c_1}\right)\mathrm{d}z + \widetilde{\zeta}_1^{(0)}, t\right), \tag{5.23}$$

where $\widetilde{\zeta}_1$ stays zero but

$$\widetilde{\zeta}_2 = \widetilde{\zeta}_1 + \int_0^t (\beta_1(z) - \beta_2(z))\,\mathrm{d}z + \widetilde{\zeta}_2^{(0)} - \widetilde{\zeta}_1^{(0)} \to -\infty, \qquad \text{as} \quad t \to +\infty \tag{5.24}$$

due to (5.22). This further gives

$$u \to \frac{2}{\sqrt{8\nu t + c_1}}\,\mathrm{sech}\, 2\widetilde{\zeta}_1, \qquad t \to +\infty. \tag{5.25}$$

Similarly, under the frame

$$\left(Y = \widetilde{\zeta}_2 = \frac{x}{\sqrt{8\nu t + c_2}} - \int_0^t \left(\beta_2(z) + \frac{4\nu}{8\nu t + c_2}\right)\mathrm{d}z + \widetilde{\zeta}_2^{(0)}, t\right), \tag{5.26}$$

which co-moves with $\widetilde{\zeta}_2$-soliton (letting $\widetilde{\zeta}_2$ stay zero), we have

$$u \to 0, \qquad t \to +\infty. \tag{5.27}$$

Thus we conclude that for the final states of two solitons involved in the degenerate case, one exists but another disappears under the condition (5.22). Obviously, similar result holds when

$$\int_0^t (\beta_1(t) - \beta_2(t))\,\mathrm{d}t \to +\infty, \qquad \text{as} \quad t \to +\infty. \tag{5.28}$$

6. Conclusion

In the paper we have investigated non-isospectral dynamics and source effects for the non-isospectral mKdVESCS-I and the non-isospectral mKdVESCS-II. One-soliton characteristics of both equations were studied and we have shown how their amplitudes and velocities rely on time. The sources do not change soliton shapes but can lead to a variety of soliton trajectories. For the non-isospectral mKdVESCS-I, its two-soliton solutions can exhibit elastic scattering with phase shifts when the sources satisfy some conditions. This fact can analytically be realized by extracting the initial and final soliton states by means of the asymptotic analysis and co-moving coordinate frames. For the non-isospectral mKdVESCS-II, its two-soliton interaction is too complicated to discuss in an analytic way, but can still scatter elastically under some conditions. For both equations, one interesting and new soliton behaviour specially related to sources is the 'ghost' solitons which appear in the degenerate two-soliton case. The typical characteristic for this case is there exists an invisible soliton involved in two-soliton interactions, and some solitons can suddenly change their sources. We note that such a degenerate case is trivial for the isospectral mKdV equation and its non-isospectral counterparts without sources. Although in the present paper we only focused on the non-isospectral mKdVESCS-I and the non-isospectral mKdVESCS-II, our discussions are general and can be generalized to other soliton equations with self-consistent sources in non-uniform media.

Acknowledgments

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References

[1] Mel'nikov V K 1986 Wave emission and absorption in a nonlinear integrable system *Phys. Lett.* A **118** 22–4

[2] Mel'nikov V K 1987 A direct method for deriving a multi-soliton solution for the problem of interaction of waves on the x, y plane *Commun. Math. Phys.* **112** 639–52

[3] Mel'nikov V K 1988 Integration method of the KdV equation with a self-consistent source *Phys. Lett.* A **133** 493–6

[4] Mel'nikov V K 1989 Capture and confinement of solitons in nonlinear integrable systems *Commun. Math. Phys.* **120** 451–68

[5] Mel'nikov V K 1989 Interaction of solitary waves in the system described by the KP equation with a self-consistent sources *Commun. Math. Phys.* **126** 201–15

[6] Mel'nikov V K 1992 Integration method of the nonlinear Schrödinger equation with a self-consistent source *Inverse Problems* **8** 133–47

[7] Leon J and Latifi A 1990 Solutions of an initial-boundary value problem for coupled nonlinear waves *J. Phys. A: Math. Gen.* **23** 1385–403

[8] Zeng Y B, Ma W X and Lin R L 2000 Integration of the soliton hierarchy with self-consistent sources *J. Math. Phys.* **41** 5453–89

[9] Lin R L, Zeng Y B and Ma W X 2001 Solving the KdV hierarchy with self-consistent sources by inverse scattering method *Physica* A **291** 287–98

[10] Zeng Y B, Ma W X and Shao Y J 2001 Two binary Darboux transformations for the KdV hierarchy with self-consistent sources *J. Math. Phys.* **42** 2113–28

[11] Zeng Y B, Shao Y J and Xue W M 2003 Negaton and positon solutions of the soliton equation with self-consistent sources *J. Phys. A: Math. Gen.* **36** 5035–43

[12] Xiao T and Zeng Y B 2004 Generalized Darboux transformations for the KP equation with self-consistent sources *J. Phys. A: Math. Gen.* **37** 7143–62

[13] Shao Y J and Zeng Y B 2005 The solutions of the NLS equations with self-consistent sources *J. Phys. A: Math. Gen.* **38** 2441–67

[14] Liu X J and Zeng Y B 2005 On the Toda lattice equation with self-consistent sources *J. Phys. A: Math. Gen.* **38** 8951–65

[15] Xiao T and Zeng Y B 2006 Bäcklund transformations for the KP and mKP hierarchies with self-consistent sources *J. Phys. A: Math. Gen.* **39** 139–56

[16] Zhang D J 2002 The N-soliton solutions for the modified KdV equation with self-consistent sources *J. Phys. Soc. Japan* **71** 2649–56

[17] Zhang D J and Chen D Y 2003 The N-soliton solutions of the sine-Gordon equation with self-consistent sources *Physica* A **321** 467–81

[18] Zhang D J 2003 The N-soliton solutions of some soliton equations with self-consistent sources *Chaos Solitons Fractals* **18** 31–43

[19] Deng S F, Chen D Y and Zhang D J 2003 The Multisoliton solutions of the KP equation with self-consistent sources *J. Phys. Soc. Japan* **72** 2184–92

[20] Gegenhasi, Integrability of a differential-difference KP equation with self-consistent sources *Preprint*

[21] Chen H H and Liu C S 1976 Solitons in nonuniform media *Phys. Rev. Lett.* **37** 693–7

[22] Hirota R and Satsuma J 1976 N-soliton solution of the KdV equation with loss and nonuniformity terms *J. Phys. Soc. Japan* **41** 2141–2

[23] Newell A C 1979 The general structure of integrable evolution equations *Proc. R. Soc.* A **365** 283–311

[24] Calogero F and Degasperis A 1978 Conservation laws for classes of nonlinear evolution equations solvable by the spectral transform *Commun. Math. Phys.* **63** 155–76

[25] Gupta M R 1979 Exact inverse scattering solution of a nonlinear evolution equation in a nonuniform media *Phys. Lett.* A **72** 420–1

[26] Burtsev S P, Zakharov V E and Mikhailov A V 1987 Inverse scattering method with variable spectral parameter *Theor. Math. Phys.* **70** 227–40

[27] Zhang D J and Chen D Y 2004 Negatons, positons, rational-like solutions and conservation laws of the KdV equation with loss and nonuniformity terms *J. Phys. A: Gen. Math.* **37** 851–65

[28] Zhang Y, Deng S F, Zhang D J and Chen D Y 2004 N-soliton solutions for the non-isospectral mKdV equation *Physica* A **339** 228–36

[29] Ning T K, Zhang D J, Chen D Y and Deng S F 2005 Exact Solutions and conservation laws for a nonisospectral sine-Gordon equation *Chaos Solitons Fractals* **25** 611–20

[30] Deng S F, Zhang D J and Chen D Y 2005 Exact solutions for the nonisospectral KP equation *J. Phys. Soc. Japan* **74** 2383–5

[31] Hirota R 1971 Exact solution of the KdV equation for multiple collisions of solitons *Phys. Rev. Lett.* **27** 1192–4

[32] Hirota R 2004 *The Direct Method in Soliton Theory (in English)* (Cambridge: Cambridge University Press)

[33] Freeman N C and Nimmo J J C 1983 Soliton solutions of the KdV and KP equations: the Wronskian technique *Phys. Lett.* A **95** 1–3

[34] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys.—JETP* **34** 62–9

[35] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform-Fourier analysis for nonlinear problems *Stud. Appl. Math.* **53** 249–315

[36] Hietarinta J 2002 Scattering of solitons and dromions *Scattering: Scattering and Inverse Scattering in Pure and Applied Science* ed R Pike and P Sabatier (London: Academic) pp 1773–91

[37] Wadati M, Sanuki H and Konno K 1975 Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws *Prog. Theor. Phys.* **53** 419–36

[38] Hirota R and Satsuma J 1981 Soliton solutions of a coupled KdV equation *Phys. Lett.* A **85** 407–8